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THE COMPLEX THREE-DIMENSIONAL ELASTOPLASTIC BENDING OF A ROD WITH A SQUARE CROSS-SECTION†

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The complex three-dimensional bending of a long rod (with a square cross-section) made of an isotropic ideally plastic and ideally cyclical material is investigated. The bending of the rod occurs due to the action of two moments, applied to its ends in such a way that the longitudinal deformation of the middle fibres of two neighbouring sides is described by a dashed line. © 1999 Elsevier Science Ltd. All rights reserved.

The fundamental equations of the theory of small elastoplastic deformations for repeated and alternating loads were formulated in [1]. This theory is an extension of Il'yushin's well-known theory of small elastoplastic deformations in the case of alternating loads. In this paper we solve a problem whose formulation is outside the scope of this theory.

1. COMPLEX THREE-DIMENSIONAL BENDING

We will investigate the complex three-dimensional bending of a long rod, the cross-section of which is a square $|x| \leqslant a$, $|y| \leqslant a$. The rod is made of an isotropic ideally plastic and ideally cyclical incompressible material. We will use a Cartesian system of coordinates (x, y, z), where the z axis is the axis of the rod while $z = \pm L$ are its ends. The rod is bent by moments M_x and M_y, applied at its ends in such a way that the longitudinal deformations of the middle filaments of two neighbouring sides ($x = -a, y = 0$) and ($x = 0, y = a$) are described by a two-section dashed line with orthogonal fragments in the deformation plane $(\varepsilon_z(-a, 0), \varepsilon_z(0, a))$, where $\varepsilon_z = \varepsilon_z(x, y)$ is the longitudinal strain.

It is clear from the condition of the problem that the stresses and strains are independent of the longitudinal coordinate.

Henceforth we will use dimensionless coordinates, referred to a, i.e. $x = \pm 1, y = \pm 1$ are the sides of the rod while $z = \pm l$ are its ends.

Taking into account the fact that the material is incompressible and isotropic, the exact solution of the problem will be sought in the form

$$\sigma_z = \sigma(x, y),\ \varepsilon = -\varepsilon^{(1)}x + \varepsilon^{(2)}y,\ \varepsilon_z = \varepsilon,\ \varepsilon_x = \varepsilon_y = -\varepsilon/2 \tag{1.1}$$

The other components of the stress and strain tensors are zero.

The mechanical meaning of the constants $\varepsilon^{(1)}$, $\varepsilon^{(2)}$ will be made clear below.

The longitudinal stresses and strains will henceforth be denoted by the quantities σ and ε, respectively.

Taking (1.1) into account, the equations of equilibrium and compatibility of the strains of the three-dimensional problem are satisfied automatically. The boundary condition in the stresses on the rod sides are also satisfied.

Moments M_x and M_y act on the cross-sections of the rod in such a way that the following deformation of the rod occurs: initially when $\varepsilon(0, 1) \equiv 0$ the deformation $\varepsilon(-1, 0)$ increases to a value $\varepsilon^{(1)}$, and the deformation $\varepsilon(0, 1)$ then increases to a value $\varepsilon^{(2)}$ for a constant value of $\varepsilon(-1, 0) \equiv \varepsilon^{(1)}$.

To fix our ideas, we will assume that $\varepsilon^{(1)} > 0$, $\varepsilon^{(2)} \geqslant 0$, where $\varepsilon^{(1)} > \varepsilon_s$ (the elastic limit for pure stretching).

We will write the relation between the stresses and strains for uniaxial variable loading (along the z axis) of an ideally plastic and ideally cyclical incompressible material. Active loading is carried out until an elastoplastic deformation ε' is built up. Unloading and the buildup of secondary plastic deformations then occur.

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For active loading we have

$$\sigma = E\varepsilon,\ \varepsilon_x = \varepsilon_y = -\varepsilon/2 \quad \text{for} \quad |\varepsilon| \leqslant \varepsilon_s$$
$$\sigma = \sigma_s \operatorname{sign}(\varepsilon),\ \varepsilon_x = \varepsilon_y = -\varepsilon/2 \quad \text{for} \quad |\varepsilon| \geqslant \varepsilon_s \tag{1.2}$$

where E is Young's modulus, ε_s is the elastic limit for stretching and $\sigma_s = E\varepsilon_s$.

For unloading from the deformed elastoplastic state ε', the following relations hold between the stresses and strain

$$\sigma - \sigma' = E(\varepsilon - \varepsilon'),\ \sigma' = \sigma_s \operatorname{sign}(\varepsilon'),\ \varepsilon_x = \varepsilon_y = -0{,}5\varepsilon$$

with $|\varepsilon - \varepsilon'| \geqslant 2\varepsilon_s$ and

$$\sigma = -\sigma_s \operatorname{sign}(\varepsilon'),\ \varepsilon_x = \varepsilon_y = -\varepsilon/2 \tag{1.3}$$

with $|\varepsilon - \varepsilon'| \geqslant 2\varepsilon_s$.

Relations (1.2) and (1.3) assume that the relation $\sigma \sim \varepsilon$ is symmetrical for compression and tension (the moduli of the elastic limits are the same for compression and tension). On the basis of these formulae, the stress σ is an odd function with respect to the origin of coordinates.

Consequently, the condition for the resultant axial force to be equal to zero is satisfied automatically.

Henceforth we will use the following notation

$$\eta = \varepsilon_s/\varepsilon^{(1)},\ \mu = \varepsilon_s/\varepsilon^{(2)}$$

At the initial stage of the solution of the problem we will determine the distribution of the axial stresses $\sigma'(x, y)$ during the first part of the process of deformation of the rod, i.e. when $\varepsilon^{(2)} = 0$.

The strain distribution has the form

$$\varepsilon' = -\varepsilon^{(1)}x$$

Active elastoplastic loading is obtained in the rod. Elastoplastic strains ε' and $\sigma' = \sigma_s$ build up in the region $\eta \geqslant x \geqslant -1$. In the elastoplastic region $\eta \leqslant x \leqslant 1$ we correspondingly have $\sigma' = -\sigma_s$. The elastic strains are distributed in the region $|x| \leqslant \eta$ and $\sigma' = E\varepsilon'$.

The following moments act over the cross-sections of the rod

$$M_x = 0 \text{ and } M_y = -2(a^3\sigma_s)\left(\frac{1}{3}\eta^2 - 1\right)$$

Since, when $\varepsilon^{(2)} = 0$, we know the strains ε' and the plastic regions, when $\varepsilon^{(2)} > 0$ we can obtain the regions of the cross-section of the rod in which active loading of the material, unloading, and the buildup of secondary plastic strains occur, as well as purely elastic loading. Then, taking relations (1.2)–(1.3) into account, we can calculate the axial stresses $\sigma(x, y)$.

In view of the fact that $\sigma(x, y)$ is odd with respect to the origin of coordinates, it is sufficient to determine the stresses in the region $y \geqslant 0$.

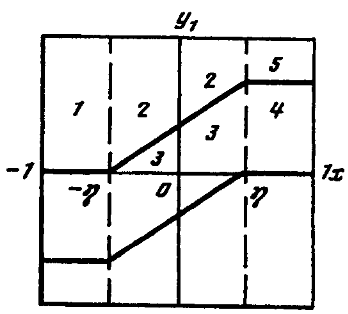

Fig. 1.

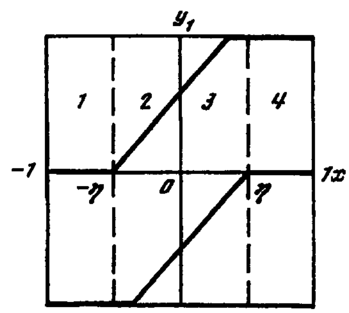

Fig. 2.

We will consider the case when $2\mu \leqslant 1$ (Fig. 1). The strains ε, ε_x, ε_y are given by (1.1). The region $y \geqslant 0$ can be conveniently split into five regions F_1, F_2, F_3, F_4, F_5 (Fig. 1).

In the region F_1, by virtue of the equation sign (ε') = sign $(\varepsilon - \varepsilon')$, active loading occurs and $\sigma = \sigma_s$. In the region F_2 loading from an elastic state occurs. Since $\varepsilon \geqslant \varepsilon_s$ we have $\sigma = \sigma_s$. The region F_3 is elastically loaded. Consequently $\sigma = E\varepsilon$. Note that the interface between the plastic region F_2 and the elastic region F_3 is described by the equation $\varepsilon_s = -\varepsilon^{(1)}x + \varepsilon^{(2)}y$. In the region F_4 elastic unloading occurs, since $|\varepsilon - \varepsilon'| \leqslant 2\varepsilon_s$, sign $(\varepsilon') = -$ sign $(\varepsilon - \varepsilon')$. Taking (1.2) into account we obtain $\sigma = -\sigma_s + E\varepsilon^{(2)}y$. In the region F_5 secondary plastic strains build up, since sign $(\varepsilon') = -$ sign $(\varepsilon - \varepsilon')$ and $|\varepsilon - \varepsilon'| \leqslant 2\varepsilon_s$. We have $\sigma = \sigma_s$. The interface between the last two regions is described by the equation $y = 2\mu$.

In the elastic region F_3 the equal-stress lines are described by the equation $-\varepsilon^{(1)}x + \varepsilon^{(2)}y = \text{const}$. In the regions F_4 and F_5 the equal-stress lines are described by the equation $y = \text{const}$.

We will write formulas for calculating the moments

$$M_x = 2(a^3\sigma_s)\left(1 - \frac{4}{3}\mu^2 + \frac{2}{3}\eta\mu^2\right)$$

$$M_y = 2(a^3\sigma_s)\mu\left(1 - \frac{1}{3}\eta^2\right)$$

Consider the case when $2\mu \geqslant 1$ (Fig. 2). This version differs from the previous one in the fact that there is no region of secondary plastic strains F_5. The formulae for determining the stresses in the regions F_1, F_2, F_3, F_4 were written down above.

The moments can be calculated from the formulae

$$M_x = 2(a^3\sigma_s)\left(\frac{1}{3} + \frac{1}{3}\eta - \frac{\eta}{8\mu}\right)\frac{1}{\mu}$$

$$M_y = -2(a^3\sigma_s)\left[\frac{1}{3}\eta^2 - 1 + \frac{1}{6}\left(\frac{\eta}{\mu}\right)^2 - \frac{1}{24\mu}\left(\frac{\eta}{\mu}\right)^2 + \frac{1}{4\mu}(1-\eta^2)\right]$$

We determine the dimensionless displacements (relative to a) from the Cauchy conditions

$$u_x = C_1 - \frac{1}{2}\varepsilon^{(2)}xy - \frac{1}{4}\varepsilon^{(1)}(y^2 - x^2) + \frac{1}{2}z^2\varepsilon^{(1)}$$

$$u_y = D_1 + \frac{1}{4}\varepsilon^{(2)}(x^2 - y^2) + \frac{1}{2}\varepsilon^{(1)}xy - \frac{1}{2}z^2\varepsilon^{(2)}$$

$$u_z = z\varepsilon$$

The displacements u_x and u_y are symmetrical with respect to the z coordinate. The constants C_1 and D_1 are calculated, taking into account the method by which the moments M_x and M_y are applied to the ends of the rod $z = \pm l$. The following condition is usually assumed

$$u_x = u_y = 0 \text{ where } z = l, x = 0, y = 0.$$

2. SIMPLE THREE-DIMENSIONAL BENDING

We will assume that moments M_x and M_y act over the cross-sections of the rod in such a way that the following deformation process occurs

$$\varepsilon^{(2)}/\varepsilon^{(1)} = \text{const} \quad \text{or} \quad \varepsilon^{(1)} = 0$$

Note, that when the last relations are taken into account, active elastoplastic loading occurs in the cross-sections of the rod, and there are no regions of unloading. The equal-stress lines are described by the equation $\varepsilon^{(1)}x + \varepsilon^{(2)}y = \text{const}$.

The strains and stresses are given by (1.1) and (1.2). Formulae for calculating the displacements were given at the end of the previous section.

This process of deformation of the rod is well known in deformed solid mechanics.

Below we give values of the dimensionless moments (with respect to the quantity $a^3\sigma_s$) M_x^*, M_y^*, $M^* = [(M_x^*)^2 + (M_y^*)^2]^{1/2}$ and M_x^{**}, M_y^{**}, M^{**} when $\varepsilon^{(1)} = 6\varepsilon_s$ and for different values of the quantities $\varepsilon^{(2)}/\varepsilon_s$ (the asterisk corresponds to the case of simple bending, while the double asterisks denote the case of complex bending)

$\varepsilon^{(2)}/\varepsilon_s$	0.00	0.50	1.25	2.25	3.00	4.00	6.00	8.00	10.0
M_x^*	0.00	0.11	0.28	0.50	0.67	0.89	1.32	1.61	1.75
M_y^*	1,98	1,98	1.95	1.89	1.81	1.69	1.32	1.00	0.80
M^*	1.98	1.98	1.97	1,95	1.93	1.91	1.87	1.90	1.93
M_x^{**}	0.00	0.38	0.91	1.52	1.73	1.85	1.93	1.96	1.98
M_y^{**}	1.98	1.74	1.36	0.88	0.66	0.50	0.33	0.25	0.20
M^{**}	1.98	1.78	1.64	1.75	1.85	1,91	1.96	1.98	1.99

The results show that the components M_x and M_y are more dependent on the deformation process than the absolute value of the moment M. A characteristic dip in the diagram of $M^{**}(\varepsilon^{(2)}/\varepsilon_s)$ with respect to the diagram of $M^*(\varepsilon^{(2)}/\varepsilon_s)$ is observed in the region of the point $\varepsilon^{(2)} = 0$. For high values of $\varepsilon^{(2)}/\varepsilon_s$ the moments M^* and M^{**} are practically identical. This effect is similar to the well-known effect of the lag of the scalar properties of plastic materials in the case of complex loading of the two-section dashed line type in Il'yushin's five-dimensional deformation space [1, 2].

REFERENCES

1. MOSKVITIN V. V., *Cyclic Loading of Structural Elements*. Nauka, Moscow, 1981.
2. IL'YUSHIN, A. A., Plasticity. Principles of the General Mathematical Theory. *Izd. Akad. Nauk SSSR*, Moscow, 1963.

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